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The inhomogeneous Toda lattice: its hierarchy and Darboux–Bäcklund transformations

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Abstract. In this paper we show how one can construct hierarchies of nonlinear differential difference equations with n-dependent coefficients. Among these equations we present explicitly a set of inhomogeneous Toda lattice equations which are associated with a discrete Schrödinger spectral problem whose potentials diverge asymptotically.

Then we derive a new Darboux transformation which allows us to get bounded solutions for the equations presented before and apply it in a specially simple case when the solution turns out to be expressed in terms of Hermite polynomials.

1. Introduction

There is an increasing number of problems in the natural sciences which are described by differential difference equations. They occur naturally in all those physical systems which are themselves discrete, like lattice systems in condensed matter, statistical mechanics and molecular physics. Moreover, differential equations for continuous systems are often reduced to difference or differential difference ones, by discretizing the underlying spacetime for reasons of technical convenience, i.e. in lattice gauge theories.

In the field of nonlinear differential difference equations a prototype example is given by the Toda lattice equation which describe the behaviour of a one-dimensional lattice consisting of N particles of unit mass interacting with their nearest neighbours through the potential

$$\phi(r) = e^{-r} + r.$$

The theory of nonlinear evolution equations on the lattice started from the pioneering work by Toda [1] in 1967 and developed in a parallel way to that of nonlinear partial differential equations; however, discrete mathematics is less developed than its infinitesimal counterpart and thus fewer results have been obtained.

The Toda lattice equation belongs to the hierarchy of equations associated with the discrete Schrödinger spectral problem

$$\psi(n-1, t; \lambda) + B(n, t)\psi(n, t; \lambda) + A(n, t)\psi(n+1, t; \lambda) = \lambda\psi(n, t; \lambda). \tag{1.1}$$

In formula (1.1), $A(n, t)$ and $B(n, t)$ are two real functions of the independent variables (n, t), where $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, λ is a complex spectral parameter and $\psi(n, t; \lambda)$, the

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wavefunction, depends on n and λ and parametrically on t. The classical theory of the Toda equation assumes that

$$\lim_{|n|\to\infty} A(n,t)-1=\lim_{|n|\to\infty} B(n,t)=0. \tag{1.2}$$

In the case of isospectral deformations of the spectral problem (1.1) with the boundary conditions (1.2) we get the class of nonlinear differential difference equations

$$\begin{pmatrix} A(n) \\ B(n) \end{pmatrix}_{,t} = \alpha(\mathscr{L},t)\begin{pmatrix} A(n)(B(n)-B(n+1)) \\ A(n-1)-A(n) \end{pmatrix} \tag{1.3}$$

where $\alpha(\mathscr{L},t)$ is an entire function with respect to its first argument and the operator $\mathscr{L}$ is given by

$$\mathscr{L}\begin{pmatrix} p(n) \\ q(n) \end{pmatrix} = \begin{pmatrix} B(n)p(n)+A(n)(q(n+1)+q(n))-A(n)S(n)(B(n+1)-B(n)) \\ p(n-1)+B(n)q(n)+A(n-1)S(n-1)-A(n)S(n) \end{pmatrix} \tag{1.4a}$$

with $S(n)$ such that

$$S(n+1)=S(n)-\frac{p(n+1)}{A(n+1)}. \tag{1.4b}$$

The simplest equation of the class is obtained by choosing $\alpha(\mathscr{L},t)$ constant: we get

$$A(n)_{,t}=A(n)(B(n)-B(n+1)) \tag{1.5a}$$

$$B(n)_{,t}=A(n-1)-A(n) \tag{1.5b}$$

where, with no restrictions, we have set $\alpha=1$. Making the substitution $A(n,t)=\exp(\chi(n,t)-\chi(n+1,t))$, $\chi(n,t)$ being a real function, we get $B(n,t)=\chi_{,t}(n,t)$ and thus

$$\chi_{,tt}(n)=\mathrm{e}^{\chi(n-1)-\chi(n)}-\mathrm{e}^{\chi(n)-\chi(n-1)} \tag{1.6}$$

i.e. the Toda lattice equation.

A first generalization of this result has been given by us a few years ago [2]; there we considered non-isospectral deformations of the spectral problem (1.1) keeping boundary conditions (1.2). In such a case λ is no longer constant, but evolves in time according to a well-defined law, namely

$$\lambda_{,t}=(\lambda^2-4)\delta(\lambda,t) \tag{1.7}$$

where δ is an entire function with respect to its first argument. The corresponding class of nonlinear differential difference equations reads

$$\begin{pmatrix} A(n) \\ B(n) \end{pmatrix}_{,t} = \alpha(\mathscr{L},t)\begin{pmatrix} A(n)(B(n)-B(n+1)) \\ A(n-1)-A(n) \end{pmatrix} + \delta(\mathscr{L},t)\begin{pmatrix} A(n)(B(n+1)(2n+3)-B(n)(2n-1)) \\ B^2(n)-4+2(n+1)A(n)-2(n-1)A(n-1) \end{pmatrix}. \tag{1.8}$$

The simplest equation in the above class is

$$A_{,t}(n)=A(n)[B(n)(1-\delta(2n-1))-B(n+1)(1-\delta(2n+3))] \tag{1.9a}$$

$$B_{,t}(n)=A(n-1)(1-2\delta(n-1))-A(n)(1-2\delta(n+1))+\delta(B^2(n)-4) \tag{1.9b}$$

for $\alpha=1$, δ constant, which corresponds, through the definition

$$A(n)=\exp[(1-\delta(2n-1))\chi(n)-(1-\delta(2n+3))\chi(n+1)]$$

to the inhomogeneous Toda lattice equation

$$\chi_{,tt}(n)=(1-2\delta(n-1))\exp(1-\delta(2n-3))\chi(n-1)-(1-\delta(2n+1))\chi(n)$$
$$-[1-2\delta(n+1)]\exp(1-\delta(2n-1))\chi(n)-(1-\delta(2n+3))\chi(n+1)$$
$$+\delta(\chi_{,t}^2(n)-4) \tag{1.10}$$

which has explicit n-dependent coefficients and corresponds to a velocity-dependent force. In both cases, (1.6) and (1.10), the solutions are obtained by solving the scattering transform for the spectral problem (1.1) with the boundary conditions (1.2).

By allowing for more general boundary conditions than (1.2), a further extension can be carried out which gives rise to new classes of nonlinear differential difference equations with variable coefficients. Of course, only certain classes of boundary conditions are admissible and the solution of the spectral problem in this case becomes generally very involved.

In the following, in section 2, we shall derive the admissible class of boundary conditions and in section 3 we provide, via the Darboux transformation, a tool to find out explicit solutions of the given nonlinear differential difference equations.

2. Construction of n-dependent Toda lattices

Let us start from (1.8), which we rewrite here adding two extra terms, obtained by taking into account that

$$\mathscr{L}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2A(n)\\B(n)\end{pmatrix} \tag{2.1a}$$

$$\mathscr{L}\begin{pmatrix}2A(n)\\B(n)\end{pmatrix}=\begin{pmatrix}A(n)[B(n+1)(2n+3)-B(n)(2n-1)]\\B^2(n)+2(n+1)A(n)-2(n-1)A(n-1)\end{pmatrix} \tag{2.1b}$$

where the operator $\mathscr{L}$ is given by (1.4):

$$\begin{pmatrix}A(n)\\B(n)\end{pmatrix}_{,t}=\alpha(\mathscr{L},t)\begin{pmatrix}A(n)(B(n)-B(n+1))\\A(n-1)-A(n)\end{pmatrix}+\beta(t)\begin{pmatrix}0\\1\end{pmatrix}+\gamma(t)\begin{pmatrix}2A(n)\\B(n)\end{pmatrix}$$
$$+\delta(\mathscr{L},t)\begin{pmatrix}A(n)(B(n+1)(2n+3)-B(n)(2n-1))\\B^2(n)-4+2(n+1)A(n)-2(n-1)A(n-1)\end{pmatrix}. \tag{2.2a}$$

The time evolution of the spectral parameter is now

$$\lambda_{,t}=\beta(t)+\gamma(t)\lambda+(\lambda^2-4)\delta(\lambda,t). \tag{2.2b}$$

In contrast to (1.8), in (2.2a) $A(n,t)$ and $B(n,t)$ do not satisfy the boundary conditions (1.2) and thus, taking into account (2.1), we are allowed to add to (1.7) $\beta(t)$ and $\gamma(t)\lambda$ and to (1.8) the terms $\beta(t)\binom{0}{1}$ and $\gamma(t)\binom{2A(n)}{B(n)}$ [2]. As we assume the boundary conditions (1.2) to be no longer valid we can introduce some reference potentials $g_1(n)$, $g_2(n)$ such that

$$A(n,t)=a(n,t)+g_1(n) \tag{2.3a}$$

$$B(n,t)=b(n,t)+g_2(n) \tag{2.3b}$$

where $a(n)$, $b(n)$ now satisfy conditions (1.2). The choice (2.3) is not compulsory: at the end of this section we shall show through an example how one can introduce other reference potentials. Our aim is to define $g_1(n)$ and $g_2(n)$ in such a way that we are still able to construct a *hierarchy* of nonlinear differential difference equations out of it. Such a requirement restricts the number of admissible choices for $g_1(n)$ and $g_2(n)$. Taking into account the fact that $(a(n), b(n))$ must satisfy the boundary conditions (1.2) and $(g_1(n), g_2(n))$ are t independent, we can write down (2.2a) as

$$\begin{pmatrix} a(n) \\ b(n) \end{pmatrix}_{,t} = f(\mathscr{L}, t)\begin{pmatrix} \alpha_0(a(n)+g_1(n))(b(n)-b(n+1))+ \\ \alpha_0(a(n-1)-a(n))+\gamma_0 b(n)+ \end{pmatrix}$$

$$\begin{pmatrix} \delta_0(a(n)+g_1(n))(b(n+1)(2n+3)-b(n)(2n-1)) \\ \delta_0(b^2(n)+2b(n)g_2(n)-4+2(n+1)a(n)-2(n-1)a(n-1)) \end{pmatrix} \tag{2.4}$$

where α_0, β_0, γ_0, δ_0, are arbitrary constants, $f(\mathscr{L}, t)$ is an entire function of its first argument and $(g_1(n), g_2(n))$ satisfy the equations

$$\alpha_0(g_2(n)-g_2(n+1))+2\gamma_0+\delta_0(g_2(n+1)(2n+3)-g_2(n)(2n-1))=0 \tag{2.5a}$$

$$\alpha_0(g_1(n-1)-g_1(n))+\beta_0+\gamma_0 g_2(n)+\delta_0(g_2^2(n)+2(n+1)g_1(n) \\ -2(n-1)g_1(n-1))=0 \tag{2.5b}$$

for $g_1(n)\neq -1$. If $g_1=-1$ then also $g_2(n)$ is constant and the solution is trivial. By comparing (2.2) and (2.4), we deduce that $\alpha(\lambda, t)=\alpha_0 f(\lambda, t)$ while $\beta(t)$, $\gamma(t)$ and $\delta(\lambda, t)$ are uniquely defined by the following identity:

$$f(\lambda, t)[\beta_0+\lambda\gamma_0+(\lambda^2-4)\delta_0]=\beta(t)+\lambda\gamma(t)+(\lambda^2-4)\delta(\lambda, t).$$

Equations (2.5), being first-order linear difference equations, can be easily solved, and we get

$$g_2(n)=\frac{1}{[\delta_0(2n+1)-\alpha_0][\delta_0(2n-1)-\alpha_0]} \\ \times\{g_2(k)[\delta_0(2k+1)-\alpha_0][\delta_0(2k-1)-\alpha_0] \\ -2\gamma_0(n-k)[\delta_0(n+k)-\alpha_0]\} \tag{2.6a}$$

$$g_1(n)=\frac{1}{(2n\delta_0-\alpha_0)[2(n+1)\delta_0-\alpha_0]} \\ \times\Bigg\{g_1(k)(2k\delta_0-\alpha_0)[2(k+1)\delta_0-\alpha_0] \\ -(n-k)\left(\beta_0\frac{\gamma_0^2}{4\delta_0}\right)[\delta_0(n+k)+\delta_0-\alpha_0]c^2 \\ \times\left(\frac{1}{[2(n+1)\delta_0-\alpha_0]^2}-\frac{1}{[2(k+1)\delta_0-\alpha_0]^2}\right)\Bigg\} \qquad (\delta_0\neq 0) \tag{2.6b}$$

where

$$c=\tfrac{1}{2}\left\{g_2(k)[\delta_0(2k+1)-\alpha_0][\delta_0(2k-1)-\alpha_0]+2\gamma_0\delta_0 k^2-2\gamma_0\alpha_0 k+\frac{\gamma_0}{2\delta_0}(\alpha_0^2-\delta_0^2)\right\}$$

and k an arbitrary integer number.

Let us now analyse some simple choices of $(\alpha_0, \beta_0, \gamma_0, \delta_0, g_1(k), g_2(k), k)$ which gives interesting reference potentials:

$$\text{(a)}\qquad \gamma_0=\delta_0=k=g_1(0)=g_2(0)=0 \qquad \alpha_0=\beta_0$$
$$g_1(n)=n \qquad g_2(n)=0 \tag{2.7}$$

$$\text{(b)}\qquad \delta_0=k=g_1(0)=g_2(0)=0 \qquad \alpha_0=2\gamma_0 \qquad \alpha_0=-4\beta_0$$
$$g_1(n)=\tfrac{1}{4}n^2 \qquad g_2(n)=n \tag{2.8}$$

$$\text{(c)}\qquad \alpha_0=\beta_0=\gamma_0=k=0 \qquad g_2(0)=-1 \qquad g_1(0)=-\tfrac{1}{4}$$
$$g_1(n)=-\frac{1}{4(2n+1)^2} \qquad g_2(n)=\frac{1}{4n^2-1}. \tag{2.9}$$

In correspondence with each one of these reference potentials we can construct a hierarchy of nonlinear differential difference equations given by (2.4). As an example, we shall write down the simplest ones, i.e. the Toda-like equations.

In case (a) we get

$$a_{,t}(n)=\alpha_0(a(n)+n)(b(n)-b(n+1)) \tag{2.10a}$$

$$b_{,t}(n)=\alpha_0(a(n-1)-a(n)). \tag{2.10b}$$

Equation (2.10), defining $a(n)=(n+1)\,\mathrm{e}^{\chi(n)-\chi(n+1)}-n$ and thus $\chi_{,t}(n)=\alpha_0 b(n)$, can be also written as

$$\chi_{,tt}(n)=\alpha_0^2[n\,\mathrm{e}^{\chi(n-1)-\chi(n)}-(n+1)\,\mathrm{e}^{\chi(n)-\chi(n+1)}+1]. \tag{2.11}$$

Equation (2.11) is associated with the spectral problem

$$\psi(n-1,t;\lambda)+b(n,t)\psi(n,t;\lambda)+(a(n,t)+n)\psi(n,t;\lambda)=\lambda\psi(n,t;\lambda)$$

where λ evolves according to the equation

$$\lambda(t)=\alpha_0 t+\lambda_0 \tag{2.12}$$

λ_0 being an arbitrary complex parameter.

In case (b) we get

$$a_{,t}(n)=\alpha_0(a(n)+\tfrac{1}{4}n^2)(b(n)-b(n+1)) \tag{2.13a}$$

$$b_{,t}(n)=\alpha_0(a(n-1)-a(n))+\tfrac{1}{2}\alpha_0 b(n). \tag{2.13b}$$

Defining $a(n)=(\frac{1}{4}n^2+1)\,\mathrm{e}^{\chi(n)-\chi(n+1)}-\frac{1}{4}n^2$, in such a way that $\chi_{,t}(n)=\alpha_0 b(n)$, expressions (2.13) yield

$$\chi_{,tt}(n)=\alpha_0^2\Big\{[\tfrac{1}{4}(n-1)^2+1]\,\mathrm{e}^{\chi(n-1)-\chi(n)}-(\tfrac{1}{4}n^2+1)\,\mathrm{e}^{\chi(n)-\chi(n+1)}$$
$$+\tfrac{1}{2}n-\tfrac{1}{4}+\frac{1}{2\alpha_0}\chi_{,t}(n)\Big\}. \tag{2.14}$$

The associated spectral problem reads

$$\psi(n-1,t;\lambda)+(b(n)+n)\psi(n,t;\lambda)+(a(n)+\tfrac{1}{4}n^2)\psi(n+1,t;\lambda)=\lambda\psi(n,t;\lambda) \tag{2.15}$$

with $\lambda(t)=\frac{1}{2}+(\lambda_0-\frac{1}{2})\,\mathrm{e}^{\alpha_0 t/2}$.

In case (c) we have

$$a_{,t}(n)=\delta_0\left(a(n)-\frac{1}{4(2n+1)^2}\right)[b(n+1)(2n+3)-b(n)(2n-1)] \tag{2.15a}$$

$$b_{,t}(n)=\delta_0\left(b^2(n)+\frac{2b(n)}{4n^2-1}-4+2(n+1)a(n)-2(n-1)a(n-1)\right) \tag{2.15b}$$

which, defining

$$a(n)=\left(1-\frac{1}{4(2n+1)^2}\right)\mathrm{e}^{(2n+3)\chi(n+1)-(2n-1)\chi(n)}+\frac{1}{4(2n+1)^2}$$

in such a way that $\chi_{,t}(n)=\delta_0 b(n)$, can be cast in the form

$$\begin{aligned}\chi_{,tt}(n)=\delta_0^2\Bigg[&\frac{(\chi_{,t}(n))^2}{\delta_0^2}+2(n+1)\left(1-\frac{1}{4(2n+1)^2}\right)\mathrm{e}^{(2n+3)\chi(n+1)-(2n-1)\chi(n)}+\frac{2\chi_{,t}(n)}{\delta_0(4n^2-1)}\\&-4+\frac{1}{(4n^2-1)^2}-2(n-1)\left(1-\frac{1}{4(2n-1)^2}\right)\mathrm{e}^{(2n+1)\chi(n)-(2n-3)\chi(n-1)}\Bigg].\end{aligned} \tag{2.16}$$

The associated spectral problem reads

$$\begin{aligned}\psi(n-1,t;\lambda)&+\left(b(n)+\frac{1}{4n^2-1}\right)\psi(n,t;\lambda)+\left(a(n)-\frac{1}{4(2n+1)^2}\right)\psi(n+1,t;\lambda)\\&=\lambda\psi(n,t;\lambda)\end{aligned} \tag{2.17}$$

with $\lambda(t)=2\coth(2\delta_0(t-t_0))$, t_0 being an arbitrary real constant.

Let us replace definition (2.2) by a new definition of the relation between the reference potentials and the old potential, namely

$$A(n,t)=a(n,t)g_1(n) \tag{2.18a}$$

$$B(n,t)=b(n,t)+g_2(n) \tag{2.18b}$$

where, as before, $a(n,t)$ and $b(n,t)$ satisfy the boundary conditions (1.2). For the sake of simplicity, let us just derive the simplest equation of the class corresponding to this new choice. It reads

$$\begin{aligned}\begin{pmatrix}a(n)\\b(n)\end{pmatrix}_{,t}=\Bigg(&\alpha_1[a(n)(b(n)-b(n+1))]\\&\alpha_1[(a(n-1)-1)g_1(n-1)-(a(n)-1)g_1(n)]\\&+\delta_1a(n)(b(n+1)(2n+3)-b(n)(2n-1))\\&+\gamma_1b(n)+\delta_1\{b^2(n)+2b(n)g_2(n)+2(n+1)(a(n)-1)g_1(n)\\&-2(n-1)(a(n-1)-1)g_1(n-1)]\Bigg).\end{aligned} \tag{2.19}$$

The corresponding equations for $g_1(n)$ and $g_2(n)$ are exactly equal to those presented before (see formulae (2.4)), with $\alpha_0=\alpha_1$, $\beta_0=\beta_1-4\delta_1$, $\gamma_0=\gamma_1$, $\delta_0=\delta_1$. Thus, for example, in case (a) we get

$$\chi_{,tt}(n)=\alpha_1^2[(n-1)\,\mathrm{e}^{\chi(n-1)-\chi(n)}-n\,\mathrm{e}^{\chi(n)-\chi(n+1)}+1]$$

with $a(n,t)=\mathrm{e}^{\chi(n,t)-\chi(n+1,t)}$; the corresponding spectral problem reads

$$\psi(n-1,t;\lambda)+b(n,t)\psi(n,t;\lambda)+a(n,t)n\psi(n,t;\lambda)=\lambda\psi(n,t;\lambda)$$

$$\lambda(t)=\alpha_1t+\lambda_0.$$

This specific example is not exceptional; indeed, one can show, in general, that the nonlinear differential difference equations one gets starting from (2.19) are not essentially different from the previous ones.

3. Darboux and Bäcklund transformations for the discrete Schrödinger spectral problem

To get the Darboux and Bäcklund transformations for the spectral problem (1.1), with the boundary conditions (2.3), we apply the standard dressing method as introduced, for discrete equations, by Bruschi *et al* [3]. For this purpose, it is appropriate to write down the spectral problem in matrix form:

$$\Phi(n-1,t;\lambda)=\begin{pmatrix}\lambda-(\chi_{,t}(n)/\alpha_0)-g_2(n) & -e^{\chi(n,t)}/G(n-1)\\ G(n-1)\,e^{-\chi(n,t)} & 0\end{pmatrix}\Phi(n,t;\lambda)$$
$$\equiv \mathcal{U}[\chi(n,t),\lambda]\Phi(n,t;\lambda) \tag{3.1}$$

where $G(n)$ is such that

$$G(n)=(1+g_1(n))G(n-1)$$

and $\Phi(n,t;\lambda)$ is given by

$$\Phi(n,t;\lambda)=\begin{pmatrix}\psi_1(n,t;\lambda) & \psi_2(n,t;\lambda)\\ G(n)\,e^{-\chi(n+1,t)}\psi_1(n+1,t;\lambda) & G(n)\,e^{-\chi(n+1,t)}\psi_2(n+1,t;\lambda)\end{pmatrix}$$

in terms of two independent solutions $\psi_1(n,t;\lambda)$ and $\psi_2(n,t;\lambda)$ of (1.1) with $A(n,t)$ and $B(n,t)$ given by (2.3). The representation (3.1) of (1.1) is such that the corresponding Hilbert–Riemann problem has canonical normalization [4] and thus the 'adding one soliton' Darboux $\mathcal{D}(n,t;\lambda)$ matrix takes the form

$$\mathcal{D}(n,t;\lambda)=\mathbb{I}+\frac{\lambda_1-\mu_1}{\lambda-\lambda_1}\mathcal{P}(n,t) \tag{3.2}$$

where λ_1, μ_1 are two complex functions, satisfying the same differential equation as λ, corresponding to two fixed different initial conditions, $\mathbb{I}$ is the 2×2 identity matrix and $\mathcal{P}(n,t)$ is a λ-*independent* 2×2 projection matrix, which can be written as

$$\mathcal{P}(n,t)=\frac{|u(n,t)\rangle\langle v(n,t)|}{\langle v(n,t)|u(n,t)\rangle} \tag{3.3}$$

where

$$|u(n,t)\rangle=\Phi(n,t;\mu_1)|u_0\rangle$$
$$\langle v(n,t)|=\langle v_0|\Phi^{-1}(n,t;\lambda_1)$$

$\Phi(n,t;\lambda)$ being a fundamental solution of (3.1) and $|u_0\rangle$, $|v_0\rangle$ two arbitrary constant vectors. Taking into account that the matrix wavefunction $\tilde{\Phi}(n,t;\lambda)$, associated with the 'dressed' potential $\tilde{\chi}(n,t)$, is given by

$$\tilde{\Phi}(n,t;\lambda)=\mathcal{D}(n,t;\lambda)\Phi(n,t;\lambda)$$

and the corresponding $\mathcal{U}$ matrix reads

$$\mathcal{U}[\tilde{\chi}(n,t);\lambda]=\mathcal{D}(n-1,t;\lambda)\,\mathcal{U}[\chi(n,t),\lambda]\mathcal{D}^{-1}(n,t;\lambda)$$

by direct calculation we can state the following 'two-parameter' Darboux theorem:

Given the discrete Schrödinger equation

$$\psi(n-1,t;\lambda)+\left(\frac{\chi_{,t}(n)}{\alpha_0}+g_2(n)-\lambda\right)\psi(n,t;\lambda)$$
$$+(1+g_1(n))\,e^{\chi(n)-\chi(n+1)}\psi(n+1,t;\lambda)=0 \tag{3.4}$$

we can construct a new potential

$$\tilde{\chi}(n)=\chi(n)-\ln\left(1+(\lambda_1-\mu_1)\frac{\psi(n;\lambda_1)\psi(n;\mu_1)}{\psi(n-1;\mu_1)\psi(n;\lambda_1)-\psi(n-1;\lambda_1)\psi(n;\mu_1)}\right) \tag{3.5}$$

and a new wavefunction

$$\tilde{\psi}(n;\lambda)=\psi(n;\lambda)+\frac{\lambda_1-\mu_1}{\lambda-\lambda_1}\psi(n;\mu_1)$$
$$\times\left(\frac{\psi(n;\lambda_1)\psi(n+1;\lambda)-\psi(n+1;\lambda_1)\psi(n;\lambda)}{\psi(n;\lambda_1)\psi(n+1;\mu_1)-\psi(n+1;\lambda_1)\psi(n;\mu_1)}\right) \tag{3.6}$$

such that (3.4) is satisfied by the dressed potential $\tilde{\chi}(n,t)$ and wavefunction $\tilde{\psi}(n,t;\lambda)$.

Thus, given a potential $\chi(n,t)$ and a wavefunction $\psi(n,t;\lambda)$ solving (3.4) the Darboux transformation provides us with a new potential $\tilde{\chi}(n,t)$ and new wavefunction $\tilde{\psi}(n,t;\lambda)$, which again satisfy (3.4).

From the two-parameter Darboux transformation (3.5), (3.6), one can get a one-parameter Darboux transformation by letting $\lambda_1\to\mu_1\equiv\mu$. In such a case we have

$$\tilde{\chi}(n)=\chi(n)-\ln\left(1+\frac{\psi^2(n;\mu)}{\psi(n-1;\mu)\psi_{,\mu}(n;\mu)-\psi_{,\mu}(n-1;\mu)\psi(n;\mu)}\right) \tag{3.7}$$

$$\tilde{\psi}(n;\lambda)=\psi(n;\lambda)+\frac{1}{\lambda-\mu}\psi(n;\mu)$$
$$\times\left(\frac{\psi(n;\mu)\psi(n+1;\lambda)-\psi(n+1;\mu)\psi(n;\lambda)}{\psi_{,\mu}(n;\mu)\psi(n+1;\mu)-\psi_{,\mu}(n+1;\mu)\psi(n;\mu)}\right). \tag{3.8}$$

Let us stress that the Darboux transformation (3.7), (3.8) depends only on one complex time-dependent parameter μ, and it does not correspond anymore to a Darboux matrix of type (3.3). In fact in this limiting case the projection matrix becomes, in whole generality, a nilpotent matrix. Consequently, we refer to (3.7), (3.8) as the 'new Darboux transformation' [5].

The corresponding Bäcklund transformation can be obtained by eliminating the wavefunction $\psi(n,t;\lambda)$ between (3.7) and the corresponding equation for $\tilde{\chi}_{,t}(n)$. It reads

$$(e^{\chi(n)}-e^{\tilde{\chi}(n)})(e^{-\tilde{\chi}(n+1)}-e^{-\chi(n+1)})\left(\left(\mu-\frac{\chi_{,t}(n)}{\alpha_0}-g_2(n)\right)(1-e^{\chi(n)-\tilde{\chi}(n)})\right.$$
$$\left.+\frac{\chi_{,t}(n)}{\alpha_0}-\frac{\tilde{\chi}_{,t}(n)}{\alpha_0}\right)^2$$
$$=4(1+g_1(n))\sinh^2(\chi(n)-\tilde{\chi}(n))\,e^{2\chi(n)}(e^{-\chi(n+1)}-e^{-\tilde{\chi}(n+1)})^2. \tag{3.9}$$

Equation (3.9) shows that the class of potentials $\chi(n, t)$ such that

$$\lim_{|n|\to\infty} \chi(n, t) = c \tag{3.10a}$$

$$\lim_{|n|\to\infty} \chi_{,t}(n, t) = 0 \tag{3.10b}$$

where c is an n-independent real quantity, is preserved by the new Darboux transformation. This is not the case for the classical Darboux transformation. The corresponding

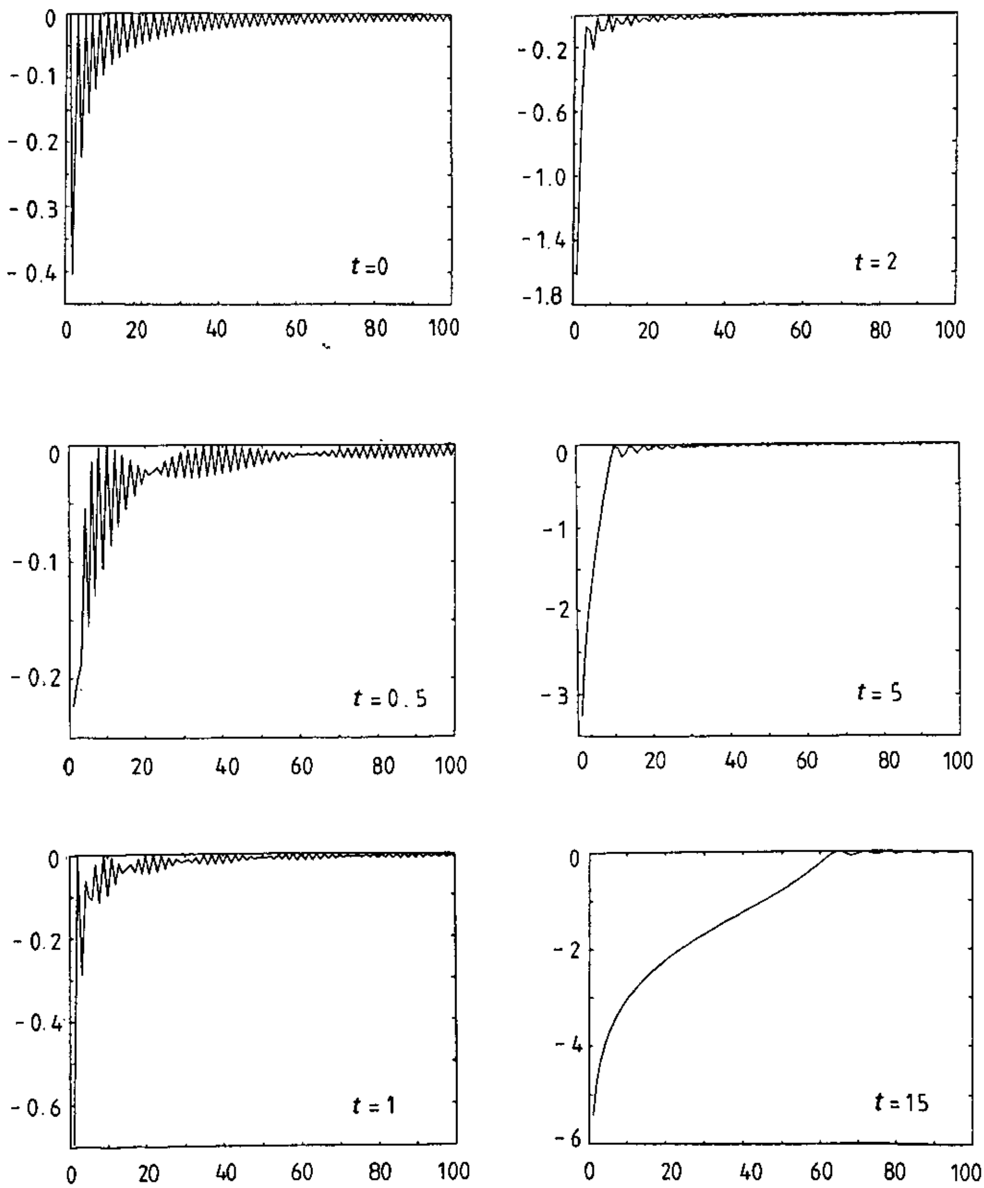

Figure 1. Plot of the function $\chi(n, t)$ at $t = 0$, $t = 0.5$, $t = 1$, $t = 2$, $t = 5$ and $t = 15$ for $\mu_0 = 0$ and $\alpha_0 = 1$.

Bäcklund transformation (see [6]) reads

$$\tilde{\chi}_{,t}(n)-\chi_{,t}(n)=\alpha_0[(1+g_1(n-1))\,e^{\tilde{\chi}(n-1)-\chi(n)}-(1+g_1(n))\,e^{\tilde{\chi}(n)-\chi(n+1)}] \qquad (3.11a)$$

$$\tilde{\chi}_{,t}(n)-\chi_{,t}(n+1)=\alpha_0(g_2(n+1)-g_2(n)+e^{\chi(n+1)-\tilde{\chi}(n+1)}-e^{\chi(n)-\tilde{\chi}(n)}). \qquad (3.11b)$$

In fact, (3.11) preserve the asymptotic behaviour (3.10) only for g_1, g_2 n-independent.

A particularly interesting case is the inhomogeneous Toda lattice given by (2.11), which corresponds to choosing $g_1(n)=n$ and $g_2(n)=0$. In this case a solution of (3.4) is given by

$$\chi(n)=0 \qquad (3.12)$$

$$\psi(n,t;\lambda)=\begin{cases}0 & n<0\\ H_n(\lambda/2^{1/2})/(2^{n/2}n!) & n\geqslant 0.\end{cases} \qquad (3.13)$$

It is worthwhile noticing that (3.13) is time dependent, through λ, and solves the whole Lax pair associated with (2.11). The application of the new Darboux transformation provide us with a non-trivial solution to the inhomogeneous Toda lattice (2.11), given by

$$\tilde{\chi}(n,t)=\begin{cases}-\ln\left(1+\dfrac{1}{2n}\dfrac{1}{nR^2(n)-\mu/2^{1/2}R(n)+\frac{1}{2}}\right) & n>0\\ -\infty & n=0\\ 0 & n<0\end{cases} \qquad (3.14)$$

where $R(n)=[H_{n-1}(\mu/2^{1/2})/H_n(\mu/2^{1/2})]$ and, according to (2.11b), we have:

$$\mu=\alpha_0 t+\mu_0 \qquad (3.15)$$

where μ_0 is an arbitrary real constant.

In figure 1 we have plotted the function $\chi(n,t)$, given by (3.14), for different values of the time t in the special case $\mu_0=0$, $\alpha_0=1$. By analysing (3.14) one can see that the zeros of $\chi(n,t)$ are just the n zeros of the Hermite polynomials $H_n(t/2^{1/2})$ and they are symmetric with respect to $t=0$. From figure 1 we observe that the 'position', say t_n, of the largest zero of $\chi(n,t)$, i.e. the furthest from the origin, is a monotonically increasing function of n. Hence the larger t is, the wider is the fraction of particles which are excited. Eventually, in the limit $t\to\infty$, $\chi(n,t)$ diverges for any n (all the particles are excited). The opposite phenomenon occur as t goes to zero.

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